UPPSALA UNIVERSITET

# Sequential Monte Carlo Methods

## Lecture 17 – SMC for Probabilistic Programs

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# Outline – Lecture 17

**Aim:** Introduce probabilistic programming as a new and exciting way of probabilistic modeling and inference, explain the basic concepts and demonstrate how SMC can be used for inference.

Outline:

1. Probabilistic programming
2. Inference (importance sampling and SMC)

# Probabilistic programming

# Motivation

Probabilistic modeling in a nutshell:

1. Write down the model in the language of mathematics
2. Derive a (bespoke) inference algorithm
3. Implement the algorithm as a computer program and run it

Developing probabilistic models and inference algorithms is a time-consuming and error-prone process.

# Probabilistic programming

Developing probabilistic models and inference algorithms is a time-consuming and error-prone process.

*Probabilistic programming* is a (relatively) new approach to change this:

- The generative model is written as a computer program
- Automatic inference (as an integral part of the programming language)

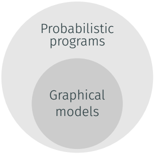
# Why probabilistic programming?

## Fast development of models

- Easy to write generative models as programs
- No need for deriving and implementing a bespoke inference algorithm

## More expressive models

- Programs can use stochastic branching and recursion, and therefore more expressive than graphical models




## Development of widely-applicable inference algorithms

# Toy model I

Consider the following toy model:

$$x_1 \sim \mathcal{N}(0, 5^2)$$
$$y_1 \sim \mathcal{N}(x_1, 1)$$
$$x_2 \sim \mathcal{N}(0.5x_1, 1)$$
$$y_2 \sim \mathcal{N}(x_2, 1)$$

The model in Matlab:

```
x1 = normrnd(0, 5);
y1 = normrnd(x1, 1);
x2 = normrnd(0.5*x1, 1);
y2 = normrnd(x2, 1);
fprintf('x1=%f, x2=%f, y1=%f, y2=%f\n', x1, x2, y1, y2);
```

We can run the script multiple times to get samples from the joint distribution $p(x1, x2, y_1, y_2)$.

# Toy model II

```
x1 = normrnd(0, 5);
y1 = normrnd(x1, 1);
x2 = normrnd(0.5*x1, 1);
y2 = normrnd(x2, 1);
fprintf('x1=%f, x2=%f, y1=%f, y2=%f\n', x1, x2, y1, y2);
```

We observed that $y_1 = 4.78$ and $y_2 = 3.12$, and want to know the posterior distribution $p(x_1 | y_1 = 4.78, y_2 = 3.12)$.

Can we change the Matlab script to get samples from this posterior distribution?

# Probabilistic programming languages

A probabilistic programming language (PPL) is a programming language that provides ergonomic support for random variables and automatic inference.

Probabilistic constructs:

- assume – declaring a random variable by specifying its probability distribution:

  $variable \sim \mathrm{Distribution}(\ldots)$

- observe – conditioning on the observed data:

  **observe** $\mathrm{Distribution}(\ldots)\ value$

# Stochastic branching and recursion I

Programs may use random variables as though any ordinary variable, even to control the flow of the execution.

Example:

$x \sim \text{Normal}(0, 1)$
**if** $x > 0.5$ **then**
    $y \sim \text{Normal}(x, 1)$
**else**
    $y \sim \text{Exponential}(1)$
**end if**

# Stochastic branching and recursion II

Example: Birth-death model for generating trees (birth rate $\lambda$, death rate $\mu$)

**function** TREE($\tau$)
    $\Delta \sim \text{Exponential}(\lambda + \mu)$
    $\tau' \leftarrow \tau - \Delta$
    **if** $\tau' < 0$ **then**
        **return** $(0, \varnothing)$
    **end if**
    $b \sim \text{Bernoulli}\left(\lambda/(\lambda + \mu)\right)$
    **if** $b$ **then**
        **return** $(\tau', \{\text{TREE}(\tau'), \text{TREE}(\tau')\})$
    **else**
        **return** $(\tau', \varnothing)$
    **end if**
**end function**

# Toy model as a probabilistic program

$y_1 \leftarrow 4.78$
$y_2 \leftarrow 3.12$

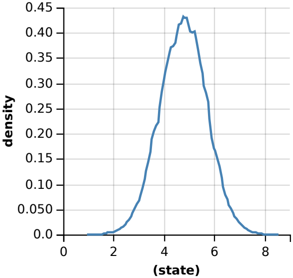
$x_1 \sim \text{Normal}(0, 5)$
**observe** $\text{Normal}(x_1, 1)\ y_1$
$x_2 \sim \text{Normal}(0.5 * x_1, 1)$
**observe** $\text{Normal}(x_2, 1)\ y_2$
**return** $x_1$

Note: A probabilistic program encodes a posterior distribution.

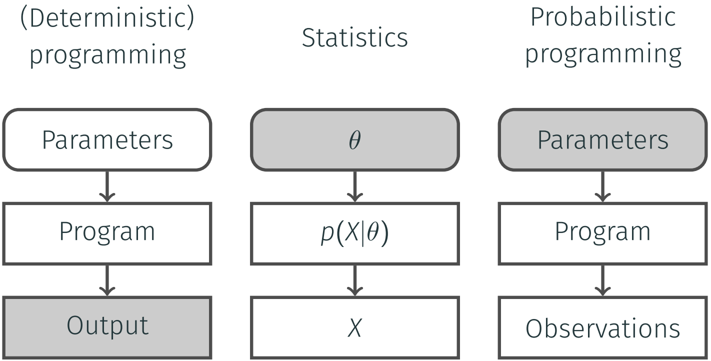
Let's implement this model in WebPPL, a simple PPL you can run in your browser (http://webppl.org).

# Toy model in WebPPL

```
var model = function() {
  var x1 = sample(Gaussian({mu: 0, sigma: 5}));
  observe(Gaussian({mu: x1, sigma: 1}), 4.78);
  var x2 = sample(Gaussian({mu: 0.5*x1, sigma: 1}));
  observe(Gaussian({mu: x2, sigma: 1}), 3.12);
  return x1;
}
var dist = Infer({method: 'SMC', particles: 10000}, model);
viz.auto(dist);
```



# Deterministic programming vs. statistics vs. PPL




*Based on a figure by Frank Wood.*

# Inference

# Inference algorithms

Automatic inference is a difficult task.

- Exact inference
  - Analytical solutions (e.g. Kalman filtering)
  - Enumeration (for discrete models of limited dimension)

- Approximate inference
  - Monte Carlo inference
    - Markov chain Monte Carlo (MCMC)
    - Sequential Monte Carlo (SMC)
    - Hamiltonian Monte Carlo (HMC)
  - Variational inference

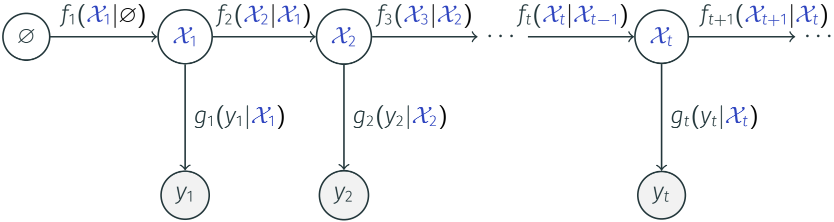
# Program state, sampling

*Note: We make some simplifications for pedagogical reasons.*

Program state: the contents of the memory used by the program to store its data.

Immediate sampling: During execution of a probabilistic program, whenever the program encounters an unobserved random variable (i.e. an assume statement), it immediately samples its value from the distribution associated with it.

# Graphical model of the execution




$\mathcal{X}_i$ denotes the state at the $i$-th observe statement.

$$p(\mathcal{X}_{1:T}, y_{1:T}) = \prod_{t=1}^{T} f_t(\mathcal{X}_t|\mathcal{X}_{t-1})g_t(y_t|\mathcal{X}_t),$$

where $\mathcal{X}_0 = \varnothing$. We are interested in the posterior probability

$$p(\mathcal{X}_{1:T}|y_{1:T}) = \frac{p(\mathcal{X}_{1:T}, y_{1:T})}{p(y_{1:T})} \propto p(\mathcal{X}_{1:T}, y_{1:T}).$$

# Importance sampling I

We can use importance sampling (IS) to sample from $p(\mathcal{X}_{1:T}|y_{1:T})$.

Proposal distribution:

$$q(\mathcal{X}_{1:T}) = \prod_{t=1}^{T} f_t(\mathcal{X}_t|\mathcal{X}_{t-1}).$$

The importance weight:

$$w(\mathcal{X}_{1:T}) = \prod_{t=1}^{T} g_t(y_t|\mathcal{X}_t).$$

**Exercise**

How can we sample from $f_t(\mathcal{X}_t|\mathcal{X}_{t-1})$?
How can we calculate $g_t(y_t|\mathcal{X}_t)$?

# Importance sampling II

Run the program forward with the following handling of probabilistic statements (checkpoints):

- assume: sample a value of the random variable (immediate sampling)
- observe: update the weight by multiplying it with the likelihood of the observed value w.r.t. its distribution and parameters (calculated during the execution).

# Importance sampling for the toy example

```
var model = function() {
  var x1 = sample(Gaussian({mu: 0, sigma: 5}));
  observe(Gaussian({mu: x1, sigma: 1}), 4.78);
  var x2 = sample(Gaussian({mu: 0.5*x1, sigma: 1}));
  observe(Gaussian({mu: x2, sigma: 1}), 3.12);
  return x1;
}
```

$w \leftarrow 1$ (initial weight)

# Importance sampling for the toy example

```
var model = function() {
  var x1 = sample(Gaussian({mu: 0, sigma: 5}));
  observe(Gaussian({mu: x1, sigma: 1}), 4.78);
  var x2 = sample(Gaussian({mu: 0.5*x1, sigma: 1}));
  observe(Gaussian({mu: x2, sigma: 1}), 3.12);
  return x1;
}
```

Sampling $x_1$ from $\mathcal{N}(0, 5^2)$: $x_1 \leftarrow 4.1$

# Importance sampling for the toy example

```
var model = function() {
  var x1 = sample(Gaussian({mu: 0, sigma: 5}));
  observe(Gaussian({mu: x1, sigma: 1}), 4.78);
  var x2 = sample(Gaussian({mu: 0.5*x1, sigma: 1}));
  observe(Gaussian({mu: x2, sigma: 1}), 3.12);
  return x1;
}
```

$$w \leftarrow w * \mathcal{N}(4.78 \mid 4.1, 1) = 0.3166$$

# Importance sampling for the toy example

```
var model = function() {
  var x1 = sample(Gaussian({mu: 0, sigma: 5}));
  observe(Gaussian({mu: x1, sigma: 1}), 4.78);
  var x2 = sample(Gaussian({mu: 0.5*x1, sigma: 1}));
  observe(Gaussian({mu: x2, sigma: 1}), 3.12);
  return x1;
}
```

Sampling $x_2$ from $\mathcal{N}(2.05, 1)$: $x_2 \leftarrow 2.7$

# Importance sampling for the toy example

```
var model = function() {
  var x1 = sample(Gaussian({mu: 0, sigma: 5}));
  observe(Gaussian({mu: x1, sigma: 1}), 4.78);
  var x2 = sample(Gaussian({mu: 0.5*x1, sigma: 1}));
  observe(Gaussian({mu: x2, sigma: 1}), 3.12);
  return x1;
}
```

$$w \leftarrow w * \mathcal{N}(3.12 \mid 2.7, 1) = w * 0.3653 = 0.1156$$

# Importance sampling for the toy example

```
var model = function() {
  var x1 = sample(Gaussian({mu: 0, sigma: 5}));
  observe(Gaussian({mu: x1, sigma: 1}), 4.78);
  var x2 = sample(Gaussian({mu: 0.5*x1, sigma: 1}));
  observe(Gaussian({mu: x2, sigma: 1}), 3.12);
  return x1;
}
```

Returned value 4.1, weight 0.1156

# SMC I

Recall the disadvantages of importance sampling.

Better idea is to use SMC methods.

## Bootstrap particle filter

- Start a set of $N$ parallel executions (*particles*) of the program.
- Repeat:
  - Run each particle until the next observe statement (incl. the weight calculation) and pause the execution.
  - Resample the particles and resume execution.

# SMC II

Run until observe $y_1$
observe $y_1$
Resampling
Cont. until observe $y_2$
observe $y_2$
Particles

# Examples of PPLs

There already exist quite a few PPLs today with different programming paradigms, for example:

- Functional: Anglican and Venture
- Imperative: Probabilistic C, Turing, Stan, Edward and Pyro
- Object-oriented: **Birch** (with delayed sampling)

# Want to learn more?


Noah D. Goodman and Andreas Stuhlmüller.
**The design and implementation of probabilistic programming languages.**
*Retrieved 2019-8-29 from http://dippl.org*

Jan-Willem van de Meent et al.
**An introduction to probabilistic programming.**
*arXiv preprint arXiv:1809.10756*, 2018.

Lawrence M. Murray and Thomas B. Schön.
**Automated learning with a probabilistic programming language: Birch.**
*Annual Reviews in Control*, 2018.